

# Calogero-Sutherland Model and Bulk-Boundary Correlations in Conformal Field Theory

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## Abstract

We show that, in any conformal field theory, the weights of all bulk primary fields that couple to  $N$   $\phi_{2,1}$  fields on the boundary are given by the spectrum of an  $N$ -particle Calogero-Sutherland model. The corresponding correlation function is simply related to the  $N$ -particle wave function. Applications are discussed to the minimal models and the non-unitary  $O(n)$  model.

The quantum Calogero-Sutherland (C-S) model has proved to be ubiquitous in theoretical physics. It has arisen in various ways in conformal field theory (CFT) in the past [1]. In this note, we point out a very direct connection. This was originally discovered [2] in the course of developing a multiparticle generalisation of Schramm-Loewner Evolution (SLE) [3], which, largely through the work of Lawler, Schramm and Werner (LSW) [4], has recently enlarged our perspective on conformally invariant random processes. However, the connection to the C-S model may be derived independently from SLE, using the basic principles of CFT, and it will now be presented in this way.

The set-up is as follows: suppose we have a CFT in the interior of the unit disc  $|z| < 1$ , with a conformal boundary condition, and consider in particular the correlation function

$$\langle \phi(e^{i\theta_1}) \dots \phi(e^{i\theta_N}) \Phi(0) \rangle = \langle \theta_1, \dots, \theta_N | \Phi \rangle \quad (1)$$

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of  $N$  boundary fields  $\phi$  with a single primary bulk field  $\Phi$  at the origin. (Of course, any correlation function in a simply connected region with  $N$  boundary fields and a single bulk field at an interior point may be related to this by a conformal mapping.) In the second expression we have written this correlation function in the operator formulation of CFT, using radial quantisation: here  $|\Phi\rangle$  is a highest weight state of the holomorphic and antiholomorphic Virasoro algebras, and  $|\theta_1, \dots, \theta_N\rangle$  is a state given by the action of boundary operators  $\phi(e^{i\theta_j})$  on the boundary state corresponding to the given conformal boundary condition. These states lie in an  $N$ -dimensional subspace of the full Hilbert space of the CFT.

Let us now suppose that  $\phi$  is a primary field which is degenerate at level 2: it is a  $\phi_{2,1}$  (or  $\phi_{1,2}$ ) field in the Kac classification. As shown many years ago by Belavin, Polyakov and Zamolodchikov [5], this implies that correlation functions such as (1) satisfy second-order differential equations. In this case we shall show that these imply the C-S equation.

First fix some notation: parametrise the central charge by  $c = 1 - 6(4 - \kappa)^2/4\kappa$ , so that the boundary scaling dimension of  $\phi$  is  $h_{2,1} = (6 - \kappa)/2\kappa$ , and the null vector condition is  $(L_{-2} - (\kappa/4)L_{-1}^2)|\phi_{2,1}\rangle = 0$ . Define the  $N$ -particle C-S hamiltonian with parameter  $\beta$  by

$$H_N(\beta) \equiv -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial \theta_j^2} + \frac{\beta(\beta-2)}{16} \sum_{1 \leq j < k \leq N} \frac{1}{\sin^2(\theta_j - \theta_k)/2} \quad (2)$$

and the free fermion wave function

$$\Psi_N(\theta_1, \dots, \theta_N) = \prod_{1 \leq j < k \leq N} (e^{i\theta_j} - e^{i\theta_k}) \quad (3)$$

Then (subject to suitable boundary conditions, see later) the ground state wave function of  $H_N(\beta)$  is  $|\Psi_N|^{\beta/2}$  with energy  $(\beta/2)^2 E_N^{\text{ff}}$  where  $E_N^{\text{ff}} = \frac{1}{24}N(N^2 - 1)$ .

Our main result is that if the correlation function (1) is non-vanishing, then the highest weight (bulk scaling dimension) of  $\Phi$  is given by

$$x_\Phi \equiv h_\Phi + \bar{h}_\Phi = \frac{\kappa}{N} \Lambda_N(8/\kappa) - \frac{4}{N\kappa} E_N^{\text{ff}} + \frac{h_{2,1}}{6} + \frac{c}{12} \quad (4)$$

where  $\Lambda_N(\beta)$  is some eigenvalue of  $H_N(\beta)$ . Moreover, the correlation function (1) is (up to a normalisation) equal to the corresponding eigenfunction, divided by  $|\Psi_N|^{2/\kappa}$ .

Consider the infinitesimal conformal transformation  $z \rightarrow z + \alpha(z)$ , where  $\alpha(z) = \sum_{j=1}^N b_j \alpha_j(z)$ , with

$$\alpha_j(z) = -z \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}} \quad (5)$$

where the  $b_j$  are infinitesimal parameters, initially chosen to be arbitrary. Note that this preserves the unit circle with the points  $\{e^{i\theta_j}\}$  removed, but near the origin it acts as a pure dilatation. It may be implemented by inserting into the correlation function (1) a factor  $\int_C T(z) \alpha(z) dz / 2\pi i - \int_C \bar{T}(\bar{z}) \overline{\alpha(z)} d\bar{z} / 2\pi i$ , where  $T$  and  $\bar{T}$  are the holomorphic and antiholomorphic components of the stress tensor, and  $C$  is any contour in  $|z| < 1$  which encircles the origin once in a counter-clockwise sense.

The effect of this insertion may be evaluated in two ways: first by shrinking the contour towards the origin and observing that, as  $z \rightarrow 0$ ,  $\alpha(z) \sim z + O(z^2)$ , and then using the operator product expansion (OPE)  $T(z)\Phi(0) = h_\Phi \Phi(0)/z^2 + O(z^{-1})$ , together with a similar antiholomorphic expression. This recovers the original correlation function multiplied by  $(h_\Phi + \bar{h}_\Phi) \sum_j b_j$ . It is important for this that  $\Phi$  is primary, so that the higher order terms in the expansion of  $\alpha(z)$  do not contribute.

The other way is to distort  $C$  so as to lie along the unit circle, with small semicircles excluding the points  $\{e^{i\theta_j}\}$ . The contributions from the parts of the contour on the circle vanish by virtue of the conformal boundary condition [6]. That from the semicircle around  $e^{i\theta_j}$  may be evaluated by writing  $z = e^{i\theta_j + i\zeta}$ , where  $\zeta$  is a local coordinate whose imaginary part is zero along the boundary. Expanding  $\alpha_j$  in powers of  $\zeta$ , we find, after a little algebra,  $\alpha_j = b_j(2/\zeta - \zeta/6 + O(\zeta^2))$ . Using the OPE with the stress tensor again, the effect of the infinitesimal transformation  $\alpha_j$  on  $\phi(e^{i\theta_j})$  is to generate  $-b_j(2L_{-2} - \frac{1}{6}L_0)\phi(e^{i\theta_j}) = -b_j((\kappa/2)(\partial/\partial\theta_j)^2 - \frac{1}{6}h_{2,1})\phi(e^{i\theta_j})$  (the minus sign is because  $C$  wraps around  $e^{i\theta_j}$  clockwise.) On the other hand,  $\alpha_j$  is regular at the points  $e^{i\theta_k}$  with  $k \neq j$ , so that  $\phi(e^{i\theta_k}) \rightarrow (1 + b_j \alpha'_j(e^{i\theta_k}))^{h_{2,1}} \phi(e^{i\theta_k} + b_j \alpha_j(e^{i\theta_k}))$ .

Putting together these contributions, the effect of  $\alpha_j$  on the boundary state is equivalent to

$$b_j \left[ -\frac{\kappa}{2} \frac{\partial^2}{\partial \theta_j^2} + \frac{1}{6} h_{2,1} - \sum_{k \neq j} \left( \cot \frac{\theta_k - \theta_j}{2} \frac{\partial}{\partial \theta_k} + i \cot \frac{\theta_k - \theta_j}{2} h_{2,1} - \frac{1}{2 \sin^2(\theta_k - \theta_j)/2} h_{2,1} \right) \right] \quad (6)$$

Now sum over  $j$ : the penultimate terms in the above sum to something proportional to  $\sum_j \sum_{k \neq j} (b_j + b_k) \cot(\theta_k - \theta_j)/2$ , which vanishes if we now take  $b_j = b_k$  for each pair  $(j, k)$ . The generator of the transformation, acting in the subspace of boundary states, is therefore

$$G \equiv -\frac{\kappa}{2} \sum_j \frac{\partial^2}{\partial \theta_j^2} - \sum_j \sum_{k \neq j} \cot \frac{\theta_k - \theta_j}{2} \frac{\partial}{\partial \theta_k} + \left( \sum_j \sum_{k \neq j} \frac{1}{2 \sin^2(\theta_k - \theta_j)/2} + \frac{N}{6} \right) h_{2,1} \quad (7)$$

The first two terms can be recognised as a similarity transform of  $H_N(4/\kappa)$ , up to a constant:

$$|\Psi_N|^{2/\kappa} G |\Psi_N|^{-2/\kappa} = \kappa \left[ H_N(4/\kappa) - (2/\kappa)^2 E_N^{\text{ff}} \right] + \left( \sum_j \sum_{k \neq j} \frac{1}{2 \sin^2(\theta_k - \theta_j)/2} + \frac{N}{6} \right) h_{2,1} \quad (8)$$

The penultimate term then combines with the potential term in  $H_N(4/\kappa)$  to give a Calogero-Sutherland hamiltonian at a *shifted* value of  $\beta$ , equal to  $8/\kappa$ :

$$|\Psi_N|^{2/\kappa} G |\Psi_N|^{-2/\kappa} = \kappa H_N(8/\kappa) - (4/\kappa) E_N^{\text{ff}} + \frac{N}{6} h_{2,1} \quad (9)$$

The eigenvalues and eigenvectors of  $G$  and  $H_N(8/\kappa)$  are thus simply related, and since the left eigenvalues of  $G$  are  $N(h_\Phi + \bar{h}_\Phi)$ , we get (4), except for the last term. This arises because the same insertion must be made in the partition function, which transforms non-trivially owing to the presence of a non-zero trace  $\langle \Theta \rangle$  of the stress tensor at the curved boundary [7]. However, a clearer derivation of this term may be found by considering the conformally equivalent geometry of a semi-infinite cylinder parametrised by the complex variable  $\ln z$ . The boundary is now no longer curved, but the dilatation operator becomes the generator of translations along the cylinder, which is [8]  $L_0 + \bar{L}_0 - c/12$ .

We now discuss some examples and applications of the general result. The C-S hamiltonian  $H_N$  commutes with the total momentum  $P \equiv -i \sum_j (\partial/\partial \theta_j)$ , whose eigenvalues correspond to the spin  $h_\Phi - \bar{h}_\Phi$  of the bulk primary field. The eigenvalue equation admits two possible boundary conditions on the wave function  $\psi$  as a given pair  $(j, k)$  of particles approach each other:  $\psi \propto |\theta_j - \theta_k|^\gamma$ , with  $\gamma = \beta/2$  ('fermionic') or  $1 - \beta/2$  ('bosonic'). These correspond to the values allowed by the BPZ fusion rules [5]: the correlation function behaves as  $|\theta_j - \theta_k|^{\gamma-2/\kappa}$ , which is consistent with the OPE

$$\phi_{2,1}(\theta_j) \cdot \phi_{2,1}(\theta_k) \sim |\theta_j - \theta_k|^{-2h_{2,1}} \mathbf{1} + |\theta_j - \theta_k|^{h_{3,1}-2h_{2,1}} \phi_{3,1} \quad (10)$$

with  $h_{2,1} = (6 - \kappa)/2\kappa$  and  $h_{3,1} = (8 - \kappa)/\kappa$ .

The eigenvalues of  $H_N(\beta)$  in the fermionic case are well-known [9]: they have the form  $\Lambda = \frac{1}{2} \sum_{j=1}^N k_j^2$ , where the allowed values of the quasiparticle momenta  $k_j$  satisfy  $\sum_j k_j = P$  and  $k_{j+1} - k_j = \frac{1}{2}\beta + p_j$ , with  $p_j$  a non-negative integer. The ground state in this sector has all the  $p_j = 0$ , and eigenvalue  $(\beta/2)^2 E_N^{\text{ff}}$ , which, after a little algebra, leads to a weight

$$x_N^{\text{f}} = \frac{N^2}{2\kappa} - \frac{(4 - \kappa)^2}{8\kappa} \quad (11)$$

of the corresponding bulk field.

### *The $O(n)$ model.*

The most immediate example of such a bulk field is in the non-unitary, non-minimal CFT that is supposed to represent the scaling limit of the  $O(n)$  model [10] with  $n \in [-2, 2]$ . This may be realised as a gas of non-intersecting closed loops and open curves [11], in which open curves ending on the boundary are known to be described by  $\phi_{2,1}$  fields [6]. The above result for  $x_N^{\text{f}}$  then agrees with the known value [12] for the bulk  $N$ -leg field. That is, the correlation function (1) is proportional to the probability that  $N$  non-intersecting curves connect the origin to the points  $e^{i\theta_j}$  on the boundary.

For  $N = 1$  and  $P \neq 0$  we find a weight  $x_\Phi = x_1^{\text{f}} + (\kappa/2)P^2$  with  $P$  an integer. The physical interpretation of these new primary fields is in terms of winding number states: each curve linking the origin and the boundary with winding number  $\chi$  is weighted by  $e^{iP\chi}$ .

For  $N = 2$ , in addition to the winding states with  $P \neq 0$ , there are new spinless primary fields corresponding to  $k_2 = -k_1 = \beta/4 + p$ , with weights  $x_\Phi = x_2^{\text{f}} + \frac{1}{2}\kappa p^2 + 2p$ , where  $p$  is a positive integer: these correspond in the  $O(n)$  model to excited modes of the pair of curves, confined by their mutual repulsion. In the case of a finite cylinder, or annulus, their weights give the exponents of correction terms in the correlation function. There are further  $N = 2$  primary fields corresponding to purely bosonic boundary conditions, with

$\gamma = 1 - \beta/2$ . These correspond to  $\Lambda_2 = (1 - \beta/2)^2 E_2^{\text{ff}}$ , which gives  $x_2^{\text{b}} = 0$  for all  $\kappa$ . This is consistent with the OPE (10): the two boundary fields are fusing to the identity on the boundary, which then couples to the identity field in the bulk. However, once again there are excited states in this sector, which correspond to possible new bulk primary fields with weights  $x_\Phi = \frac{1}{2}\kappa p(p+1) - 2p$ . These two types of boundary condition may be understood within the  $O(n)$  model as follows: each boundary field  $\phi_{2,1}$  and its attached curve carry an  $O(n)$  vector index. If the two labels are different, the curves cannot join, and the fusion is into a  $\phi_{3,1}$  field transforming according to a tensor representation of  $O(n)$ . However, if the labels are the same, the curves can join up before reaching the origin. The fusion in this case is into the identity field, and the fact that the leading coupling is now to a bulk primary field with  $x_\Phi = 0$  means that the probability of the curves joining, and not therefore passing through the origin, is unity. The weights corresponding to the excited states, with  $p \geq 1$ , then give the exponents of correction terms to this for an annulus.  $p = 1$  corresponds to the bulk energy density field of the  $O(n)$  model.

However, still for  $N = 2$ , there are other possible ‘mixed’ boundary conditions which are fermionic as  $\theta_2 - \theta_1 \rightarrow 0+$ , and bosonic as  $\theta_2 - \theta_1 \rightarrow 2\pi-$ . The ground state in this sector has energy  $\Lambda_2 = \frac{1}{16}$ , which gives  $x_\Phi = (3\kappa - 8)(8 - \kappa)/32\kappa$ . This is the exponent determining the relative probability that a curve, whose ends are attached at nearby points on the boundary, should enclose the origin or not. For  $\kappa = 6$ , this is the ‘one-arm’ exponent of percolation (related to the probability that the origin lies in a cluster which touches the boundary), as computed by LSW [13]. Once again, there are excited states in this sector whose energies give corrections to scaling.

#### *Minimal models.*

These correspond to rational  $\kappa = 4k/k'$ , and the allowed values of the weights of scalar bulk primary fields are given by the Kac formula  $x_{r,s} = ((4r - \kappa s)^2 - (4 - \kappa)^2)/8\kappa$  with  $1 \leq r \leq k - 1$  and  $1 \leq s \leq k' - 1$ . For these to agree with (4) imposes a severe constraint. If

it cannot be satisfied, it implies that the correlation function (1) must vanish. For example, the weight of the 1-leg field corresponds formally to  $x_{1/2,0}$ , but for this to appear in the table of allowed values it is necessary that  $k$  is odd, with  $r = (k-1)/2$ , and  $k'$  even, with  $s = k'/2$ . This is of course consistent with the fusion rules of boundary conformal field theory. For  $N = 2$ , with fermionic boundary conditions, no solution for  $(r, s)$  in the allowed range is possible, indicating that, in a minimal model, two  $\phi_{2,1}$  fields on the boundary can couple to the bulk only through fusion into the identity. In that case, coupling is allowed, as long as  $p \leq [k/2] - 1$ , because the corresponding allowed weight is  $x_\Phi = x_{1,2p+1}$ .

### *Comparison with multiple SLEs.*

Although the above CFT arguments are self-contained, it is instructive to compare them with those of Ref. [2]. There, a multi-particle generalisation of SLE was proposed in which the  $N$  curves connecting the boundary with the origin are ‘grown’ dynamically, starting from the boundary at time  $t = 0$  and reaching the origin as  $t \rightarrow \infty$ . This process is described in terms of the evolution of the conformal mapping  $g_t(z)$  which sends the simply connected region not yet excluded by the curves into the whole unit disc. This turns out to satisfy  $dg_t = \sum_j b_j \alpha_j(g_t)$ , with  $\alpha_j$  having the same form as in (5). In this picture, however, the  $\theta_j$  become functions of  $t$ , evolving, if we take  $b_j = dt$  for all  $j$ , according to Dyson’s brownian motion [14]:

$$d\theta_j = \sum_{k \neq j} \cot((\theta_j - \theta_k)/2) dt + dB_j(t) \quad (12)$$

The terms on the right hand side correspond to a mutual repulsion and a stochastic noise. For  $N = 1$  it is known [3] that this process (radial SLE) gives the correct measure on the continuum limit of a single curve. The corresponding Fokker-Planck equation for the joint probability distribution  $P(\{\theta_j\}; t)$  has the form  $dP/dt = \mathcal{L}P$ , and it was argued in Ref. [2] that the asymptotic equilibrium solution of this equation, satisfying  $\mathcal{L}P_{\text{eq}} = 0$ , should give the distribution of the points on the boundary in an equilibrium  $2d$  critical system such as

the  $O(n)$  model. This gives the result  $|\Psi_N|^{4/\kappa}$ , in contradiction with the CFT prediction  $|\Psi_N|^{2/\kappa}$ .

This discrepancy may be traced to the form of

$$\mathcal{L}^\dagger = \frac{\kappa}{2} \sum_j \frac{\partial^2}{\partial \theta_j^2} + \sum_j \sum_{k \neq j} \cot \frac{\theta_k - \theta_j}{2} \frac{\partial}{\partial \theta_k} \quad (13)$$

The first term comes from averaging over the white noise  $dB_j$ , and the second from the repulsion. Comparing this with the expression (7) for the conformal generator  $G$ , we see that the last term, proportional to  $h_{2,1}$ , is absent. This arises, in the CFT calculation, from the transformation of  $\phi(e^{i\theta_k})$  under the local scale transformation induced by the conformal mapping  $\alpha_j$ , but it is missing in the multiple SLE approach. Indeed, there is no obvious way of modifying (12) so as to incorporate such a term.

However, the argument in Ref. [2] that the joint distribution of the boundary points  $\{\theta_j\}$  in a critical system should be given by the equilibrium distribution of the process (12) was also based on the assumption that the measure on the  $N$  curves was strictly conformally invariant under  $g_t$ . For the case of a single curve with each end on the boundary of a simple connected region, this is known not to be the case in general – rather, it is conformally covariant. [4] The covariance factor is just the product of the local scale transformations at each end, raised to the power  $h_{2,1}$ . When the arguments of Ref. [2] are modified to take into account these factors, the result agrees precisely with that of the CFT argument given earlier. The fact that  $SLE_\kappa$  is now associated with  $H_N(\beta)$  with  $\beta = 8/\kappa$  is much more satisfactory. For example, it suggests that the duality of SLE [15] under  $\kappa \rightarrow 16/\kappa$  is related to the known duality of the C-S hamiltonian [16] under  $\beta \rightarrow 4/\beta$ .

To summarise, we have shown that the quantum Calogero-Sutherland model arises in a very simple way in bulk-boundary conformal field theory. The full spectrum is realised in the non-unitary CFT of the  $O(n)$  model, and it predicts the scaling dimensions of new primary fields in that theory. In minimal models, it places severe restrictions on which bulk fields can couple to the boundary. In general, because of the Galilean invariance of the C-S

hamiltonian, any scalar primary bulk operator is associated with a tower of other primaries of spin  $s$ , with the differences in conformal weights proportional to  $s^2$ . Thus, although the spectrum of Virasoro descendants in a CFT is relativistic, the spectrum of these primaries has a non-relativistic form.

Although we have considered only the case of  $\phi_{2,1}$  fields on the boundary, our arguments can be generalised to include other boundary fields corresponding to degenerate Virasoro representations. These will lead to higher-order differential operators. Similar generalisations to WZWN models are also possible.

Recently Bauer and Bernard have extended their analysis of CFT as a probe of SLE [17] to the radial case [18]. Some of their results overlap with ours.

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